

Theory of non-local point transformations - Part 1: Representation of Teleparallel Gravity

Massimo Tessarotto

*Department of Mathematics and Geosciences, University of Trieste, Italy and
Institute of Physics, Faculty of Philosophy and Science,
Silesian University in Opava, Bezručovo nám.13, CZ-74601 Opava, Czech Republic*

Claudio Cremaschini

*Institute of Physics, Faculty of Philosophy and Science,
Silesian University in Opava, Bezručovo nám.13, CZ-74601 Opava, Czech Republic
(Dated: January 18, 2016)*

In this paper the extension of the functional setting customarily adopted in General Relativity (GR) is considered. For this purpose, an explicit solution of the so-called Einstein's Teleparallel problem is sought. This is achieved by a suitable extension of the traditional concept of GR reference frame and is based on the notion of non-local point transformation (NLPT). In particular, it is shown that a solution to the said problem can be reached by introducing a suitable subset of transformations denoted here as *special NLPT*. These are found to realize a phase-space transformation connecting the flat Minkowski space-time with, in principle, an arbitrary curved space-time. The functional setting and basic properties of the new transformations are investigated.

PACS numbers: 02.40.Hw, 04.20.-q, 04.20.Cv

1 - INTRODUCTION

In this paper and in the subsequent related ones (Parts 2 and 3) the problem is investigated of the extension of the customary functional setting which lays at the basis of relativistic theories in physics. These notably include, besides classical electrodynamics, relativistic classical mechanics and relativistic quantum mechanics, in particular the so-called *Standard Formulation to General Relativity* (SF-GR), i.e., Einstein's original approach to his namesake field equations. The latter, as is well known, uniquely determine the metric tensor $g_{\mu\nu}(r)$ [1–4, 6, 7] associated with a prescribed parametrization of the physical space-time, identified with the 4–dimensional connected and time-oriented real metric space $D^4 \equiv (\mathbf{Q}^4, g)$, with $\mathbf{Q}^4 \equiv \mathbb{R}^4$. Such a functional setting is realized by the group of transformations connecting arbitrary GR-reference frames, i.e., arbitrary 4–dimensional curvilinear coordinate systems spanning the same prescribed space-time D^4 . In SF-GR this is usually identified with the group $(\{P\})$ of invertible *local point transformations* (LPT) P and its inverse P^{-1} , namely

$$P : r^\mu \rightarrow r'^\mu = r'^\mu(r), \quad (1)$$

$$P^{-1} : r'^\mu \rightarrow r^\mu = r^\mu(r'), \quad (2)$$

where the initial and transformed 4–positions r^μ and r'^μ are assumed to span the same space-time (\mathbf{Q}^4, g) . Hence, by definition, the group $\{P\}$ leaves invariant (\mathbf{Q}^4, g) , which must therefore be identified with a differential manifold. It is obvious that such a functional setting is intrinsic to SF-GR, i.e., it is actually required for the validity of SF-GR itself. The same transformations (Eqs.(1)-(2)) are assumed also to warrant the global validity of the so-called Einstein's *General Covariance Principle* (GCP) [8]. In other words, the transformations defined by Eqs.(1)-(2) must be endowed with a suitable functional setting (see related discussion in Section 2), referred to here as *LPT-functional setting*, which permits in turn also the corresponding realization of GCP. Such a principle is therefore referred to as *LPT-GCP*. In particular, this means that LPT must be smoothly differentiable so to uniquely and globally prescribe also the 4–*tensor transformation laws* of the displacement 4–vectors, namely

$$\begin{cases} dr^\mu = \mathcal{J}_\nu^\mu dr'^\nu, \\ dr'^\mu = (\mathcal{J}^{-1})_\nu^\mu dr^\nu. \end{cases} \quad (3)$$

Here \mathcal{J}_ν^μ and $(\mathcal{J}^{-1})_\nu^\mu$ denote the direct and inverse Jacobian matrices which take the so-called *gradient form*, i.e.,

$$\mathcal{J}_\nu^\mu(r') \equiv \frac{\partial r^\mu(r')}{\partial r'^\nu}, \quad (4)$$

$$(\mathcal{J}^{-1})_\nu^\mu(r) \equiv \frac{\partial r'^\mu(r)}{\partial r^\nu}, \quad (5)$$

which uniquely-globally prescribe also the corresponding 4-tensor transformation laws of *all tensor fields* which characterize SF-GR.

However, in this work we intend to show that a new approach alternative to the one adopted in GR founded on the introduction of an extended functional setting is actually possible. This is based both on mathematical and physical considerations. Starting point is the notion of *non-local point transformations (NLPT)*, and will be referred to here as *NLPT-functional setting*. Such a setting should permit, in principle, to map in each other intrinsically different space-times (\mathbf{Q}^4, g) and (\mathbf{Q}'^4, g') , i.e., space-times which cannot be otherwise connected by means of the group $\{P\}$.

The issue concerns the prescription of the appropriate class of GR-reference frames (*GR-frames*) to be adopted as well as of the transformations connecting them. It is well-known that in the customary approach to GR [1–4, 6] the GR-frames are identified with arbitrary sets of curvilinear coordinate systems, while the latter are realized by means of LPT, i.e., smoothly differentiable real maps depending locally on position only. Nevertheless, as discussed below, there exist theoretical motivations which suggest the mathematical and/or physical inadequacy (in the context of GR) equivalently either of the functional setting based on LPT only or the traditional concept of GR reference frame, which in fact relies - in turn - on the use of the same type of coordinate transformations. These motivations include a number of problem-cases of special physical relevance (see below).

In Part 1, in particular, the example-case is considered which deals with the so-called teleparallel representation of GR, also known as *Einstein Teleparallelism* or (Einstein) *Teleparallel Gravity* [9]. We intend to prove that in the context of teleparallel gravity the introduction of new types of GR-frames and coordinate transformations is mandatory. These are found to be realized respectively by means of a kind of phase-space reference frames, denoted as *extended GR-frames*, and a suitably-defined set of phase-space maps, which involve in particular the introduction of appropriate non-local coordinate transformations, identified here as *special NLPT*.

Historical ante factum and the issue of non-local generalizations of GR

An ongoing subject of theoretical investigations in GR concerns its possible non-local modifications. Recent literature investigations in this category are several. Examples can be found, for instance, in Refs.[10–15], where non-local generalizations of the Einstein theory of gravitation have been proposed. Such a kind of non-local GR models lead typically to suitably-modified forms of the Einstein equation [1] in which non-local field interactions are accounted for, in analogy with corresponding non-local features of the electromagnetic field occurring in Classical Electrodynamics.

It is well-known that the *LPT-functional setting* adopted by Einstein in his original formulation of GR is uniquely founded on the classical theory of tensor calculus on manifolds. The historical foundations of the latter, in turn, date back to the so-called absolute differential calculus developed at the end of 19th-century by Gregorio Ricci-Curbastro and later popularized by his former student and collaborator Tullio Levi-Civita [2, 5]. However, a basic issue that arises in GR and its possible non-local generalizations, as well as more generally in classical and quantum theories of particles and fields, is whether these theories themselves might exhibit possible contradictions with the validity of the LPT-GCP and consequently a more general functional setting should be actually adopted for the treatment of these disciplines.

To better elucidate the scope and potential physical relevance of the topics indicated above, it is worth to highlight in detail some of the main related issues and physical problems to be found in the literature which, as explained in detail below, are still challenging and whose solution appears of critical importance in GR. These include:

1. *Problem #1: Teleparallel approach to GR* - One of the most remarkable physical examples of violation of LPT-GCP - and the one which motivates the present paper - occurs however in the framework of the Einstein's teleparallelism (or Teleparallel problem, see Refs.[9]), and possibly also in some of its recently-proposed generalizations [16–18]. The conclusion is of immediate and patent evidence. Indeed, such a theory is intended to map in each other intrinsically different space-times. In the case of Teleparallelism one of such space-times is identified, by construction, with the flat time-oriented Minkowski space-time. As discussed below (see Section 3), this is achieved by a suitable matrix transformation (*teleparallel transformation*) between the corresponding metric tensors, denoted as teleparallel problem (TT-problem), which lies at the basis of such an approach (see Eq.(17) or equivalent Eq.(18)). A number of related issues arise which concern in particular:

- *Problem #P1₁* - The realization and possible non-uniqueness feature of the mapping to be established between the two space-times occurring in the teleparallel transformation itself. This refers in particular of what might/should be:
 - A) the actual representation of the corresponding coordinate transformations;

- B) their local and possible non-local dependences;
 - C) the possible existence/non-existence of corresponding tensor transformation laws for observable tensor fields, etc.
 - *Problem #P1₂* - The fact that obviously such problems, and the TT-problem itself, cannot be solved in the framework of the validity of the LPT-GCP.
 - *Problem #P1₃* - The physical implications of the theory, with particular reference to the explicit construction of special NLPT.
2. *Problem #2: Diagonalization of metric tensors and complex transformation approaches to GR* - A second notable example concerns the adoption in GR of complex-variable transformations, such as the so-called Newman-Janis algorithm [19–21]. This is frequently used in the literature for the purpose of investigating a variety of standard or non-standard GR black-hole solutions [22, 23], as well as alternative theories of gravitation, such as the one based on non-commutative geometry [24]. Its basic feature is that of permitting one to transform, by means of a complex coordinate transformation, a diagonal metric tensor corresponding to a spherically-symmetric and stationary configuration (like the Schwarzschild one) into a non-diagonal one corresponding to a rotating black-hole (like the Kerr solution). On the other hand, a number of issues arise concerning the Newman-Janis algorithm. These include:
- *Problem #P2₁* - First, it is complex, so that the transformed coordinates are complex too. This inhibits their objective physical interpretation in terms of physical observables.
 - *Problem #P2₂* - The fact that, as for the Teleparallel transformation, the diagonalization problem at the basis of the same transformation cannot be solved in the framework of the validity of the LPT-GCP. Indeed, the Newman-Janis algorithm seems worth to be mentioned especially in view of the fact that it obviously represents a patent violation of the LPT-GCP.
 - *Problem #P2₃* - The physical meaning of the transformation: one cannot ignore that fact that there is no clear understanding regarding its physical interpretation and ultimately as to why the algorithm should actually work at all.
 - *Problem #P2₄* - Finally, despite the obvious fact that the Teleparallel transformation provides in principle also a solution to the diagonalization problem, there is no clear connection emerging between the same transformation and the Newman-Janis algorithm.
3. *Problem #3: Acceleration effects in relativistic classical electrodynamics* - A third issue worth to be pointed out for its potential relevance in the present discussion concerns the role of acceleration on GR reference frames as discussed for example in Refs.[25, 26]. These papers deal with the necessity of taking into account, both in the context of GR and Maxwell's equations, possible acceleration-induced non-local effects. However, the precise mathematical formulation and physical mechanisms by which non-locality should manifest itself must still be fully understood.
- In fact, a number of basic issues remain unanswered. These concern in particular the following ones:
- *Problem #P3₁* - First, the precise prescription of the mathematical setting of the theory and in particular the implementation and possible functional realization of the non-local acceleration effects and the possible connection with the theory of Teleparallel gravity in the context of GR remain unclear.
 - *Problem #P3₂* - Indeed, non-local acceleration effects are introduced by postulating directly "ad hoc" integral representations (or "transformation laws") for appropriate tensor fields.
 - *Problem #P3₃* - The validity of these transformation laws, namely the reason why ultimately they should apply, and consequently their physical interpretation, remain both ultimately unclear.
4. *Problem #4: Non-local effects in classical electrodynamics* - A further intriguing example which is by itself sufficient to demonstrate the role of non-locality in physics can be found in the framework of a special-relativistic treatment of classical electrodynamics. This concerns the so-called electromagnetic radiation-reaction (EM-RR) problem, i.e., the dynamics of an extended charge in the presence of its self-generated EM field. As shown in Refs.[27, 28] such a problem can be rigorously treated in the framework of a first-principle approach based on the Hamilton variational principle. In such a context the sources of non-locality appears at once as being due to the finite size of charged particles. Indeed, its physical origin is related to the retarded EM interaction of the extended

particle with itself [29–33]. However, a further fundamental physical implication also emerges. In fact, as shown in Ref.[29], in the variational action functional *point Lorentz transformations must be considered as non-local*, thus effectively extending the class of local Lorentz transformations usually considered in special relativity. This arises because, in order to preserve the scalar property of the relativistic Lagrangian in the Hamilton variational principle, the point transformations (realized by Lorentz transformations) must act “non-locally”. In fact, in contrast to local coordinate transformations which are supposed to act only on the explicit local functional dependences, in such a case the Lorentz transformations must also act on the non-local dependences appearing in the same functional. In particular, the following issues should be answered:

- *Problem #P4₁* - First, the precise prescription of the transformation laws with respect to the group on NLPT should be achieved for the EM 4–potential A^μ and of the corresponding EM Faraday tensor $F^{\mu\nu}$.
- *Problem #P4₂* - Second, it remains to be ascertained whether and possibly under what conditions the transformations indicated above are realized by means tensor transformation laws, i.e., respectively for A^μ and $F^{\mu\nu}$, transformation laws formally identical to those determined by the 4–position infinitesimal displacement dr^μ or the dyadic tensor $dr^\mu dr^\nu$.

The key question which needs to be ascertained in the context of GR is whether these problems do actually require, as anticipated above, the introduction of a more general class of GR-reference frames. In fact, despite previous interesting but incomplete solution attempts [25, 26], a basic issue which still remains unsolved nowadays concerns the construction of the explicit general form and physically-admissible realizations which the transformations occurring among arbitrary GR-frames should take. The problem matter refers therefore to possible non-local generalization of the customary local tensor calculus and coordinate transformations to be adopted in GR. This is actually the task which we intend to undertake in the present investigation.

Under such premises it must be noted that the present work departs, while being at the same time also in some sense complementary, from the non-local GR theories indicated above. In fact it belongs to the class of studies aimed at introducing in the context of GR a new type of non-local phenomena based on the coordinate transformations established between GR-reference frames and at the same time extending the functional setting customarily adopted in such a context.

Outline of the investigation

More precisely, the overall work-plan of the investigation is to address the problem of the non-local generalization of GR achieved by a suitable extension of its functional setting. This task is by no means trivial since it concerns basic theoretical issues and physical problems which have remained unsolved to date in the literature and whose solution presented in this investigation for the first time appears of critical importance in General Relativity (GR). In detail these include:

1. *Topics #1* - The identification of possible generalizations of the LPT-setting customarily adopted in GR, based on physical example-cases. A notable problem of this type is realized by Einstein’s approach to the so-called *Einstein’s teleparallelism*. In such a context, the issue arises whether such a theory can be recovered from SF-GR by means of a suitable mathematical, i.e., purely conceptual, viewpoint. This involves the introduction of appropriate non-local point transformations (or NLPT). Their determination, despite being of basic importance in GR, still remains essentially unknown to date. It must be stressed, in this regard, that the possible prescription of NLPT is by no means “*a priori*” obvious since they remain - it must be stressed - largely arbitrary and intrinsically non-unique. For this purpose in Part 1 *Problems #P1₁–#P1₃* are addressed. Their solution is crucial for their identification. This goal can be reached based on the adoption of a suitable sub-set of NLPT, referred to here as *special NLPT-group* $\{P_S\}$ acting on appropriate *extended GR-frames* which are defined with respect to prescribed space-times. For definiteness, in view of warranting the validity of suitable tensor transformation laws for the metric tensor which is associated with the Teleparallel transformation (see Eq.(44) below), in the present treatment these transformations are assumed to preserve the line element (see Section 4 below), in other words they are required to map space-times (Q^4, g) and $(Q'^4, g') \equiv (M'^4, \eta)$ having the same line elements ds and ds' .
2. *Topics #2* - In Part 2 *Problems #P2₁–#P2₄* are addressed. For such a purpose the determination is done of the group of *general non-local point transformations* (general NLPT) connecting subsets of two generic curved space-times (Q^4, g) and (Q'^4, g') . This is referred to here as *general NLPT-group* $\{P_g\}$. The task posed here

involves also their physical interpretation based on a suitable Gedanken experiment. This refers, in particular to three distinct issues:

- A) The possible conceptual realization of a measure experiment (Gedanken experiment), simulating the action of a generic, NLPT on a GR-reference frame on the physical space-time.
 - B) The prescription of the family of NLPT, exclusively based on a suitable set of mathematical, i.e., axiomatic, prescriptions, which should be nevertheless physically realizable in principle for arbitrary GR-reference frames which are defined with respect to a prescribed (physical) space-time.
 - C) As an illustration of the theory, the explicit construction of possible physically-relevant transformations of the group $\{P_g\}$, with special reference to the problem of the *diagonalization of metric tensors* in GR.
3. *Topics #3* - The investigation of *physical implications* of the general NLPT-functional setting, with particular reference to the identification of possible *acceleration effects* in GR and classical electrodynamics. The goal of Part 3 is to look for a possible solution of *Problems #3 and #4* indicated above. This involves in particular:
- A) the investigation of the role of acceleration on GR reference frames;
 - B) the search of possible 4-tensor transformation laws occurring respectively for the 4-acceleration field and the EM 4-vector potential will be investigated, with respect to the group of NLPT $\{P_g\}$ established between suitable subsets of two arbitrary curved space-times (\mathbf{Q}^4, g) and (\mathbf{Q}'^4, g') . Regarding point B), the key related issue concerns in fact to ascertain whether and under what conditions 4-tensor transformation laws exist both for the 4-acceleration and the EM 4-vector potential.

In the present manuscript (Part 1) topics #1 will be addressed. Topics #2 and #3 will be, instead, discussed respectively in Parts 2 and 3.

Goals and structure of the paper

Given these premises, we are now in position to state in detail the structure of the present manuscript, pointing out the goals posed in each of the following sections which are accordingly listed below.

1. *GOAL #1* - The first one, discussed in Section 2, includes the task of displaying the functional setting (LPT-functional setting) usually adopted in SF-GR. Its basic features are pointed out together with some basic implications relevant in the subsequent discussion.
2. *GOAL #2* - The second one, which is presented in Section 3, concerns an insight of the Einstein's theory of teleparallelism and the related Teleparallel Problem (*TT-problem*). For this purpose its basic assumptions, formulation and implications are analyzed in detail.
3. *GOAL #3* - Based on the investigation of the same TT-problem, in Section 4 the theory of *special NLPT* is developed. It is shown that for this purpose a new *NLPT-functional setting* is required. As a consequence it is shown that a phase-space map can be established between the Minkowski flat space-time and an in principle arbitrary curved space-time. This involves, in particular, the adoption of non-local point transformation, referred to as special NLPT.
4. *GOAL #4* - In Section 5, the conditions of existence of NLPT are discussed, which yield particular solutions of the TT-problem.
5. *GOAL #5* - In Section 6, as application of the theory of special NLPT, a sample case is investigated.
6. *GOAL #6* - Finally, in Section 7 the main conclusions of the paper are drawn.

2 - THE LPT- FUNCTIONAL SETTING AND ITS IMPLICATIONS

In order to state clearly the problem and its related motivations, we first recall the functional setting which - as anticipated above - is usually adopted both in relativistic theories as well as in Einstein's 1915 theory of gravitation [1], i.e., SF-GR itself. In both cases the goal is, in principle, to predict all physically-relevant realizations of the observables. In the case of GR these concern the physical space-time itself $D^4 \equiv (\mathbf{Q}^4, g)$. As is well-known, in SF-GR

this is identified with a 4-dimensional Lorentzian metric space on $\mathbf{Q}^4 \equiv \mathbb{R}^4$ which is endowed with a prescribed metric tensor $g_{\mu\nu}(r)$ when the same set Q^4 is represented in terms of a given set of curvilinear coordinates $\{r^\mu\} \equiv r$. Nevertheless, validity of GR, and in particular of the Einstein equation itself, requires to couch them in a suitable mathematical framework.

As recently pointed out in Ref.[7] in the context of a variational treatment of SF-GR, this involves, besides the fulfillment of a suitable property of gauge invariance, also the adoption of Classical Tensor Analysis on Manifolds. In other words, as anticipated above, both GR and the same Einstein equation should embody by construction the validity of LPT-GCP, namely formulated consistent with the LPT- functional setting. This means explicitly that the following mathematical requirements (A-C) should apply:

A) All physically-observable tensor fields defined on space-time (\mathbf{Q}^4, g) must be realized by means of 4-tensor fields with respect to a suitable ensemble of coordinate transformations connecting in principle arbitrary, but suitably related, 4-dimensional curvilinear coordinate systems, referred to as *GR-reference frames*, r^μ and r'^μ .

B) The PDEs, together with their corresponding variational principles, which characterize all classical and quantum physical laws should satisfy *the criterion of manifest covariance, whereby it should be possible to cast them in all their realizations in manifest 4-tensor form.*

C) The set of coordinate transformations indicated above is identified with the group of transformations that in Eulerian form are prescribed by means of the invertible maps (1)-(2) which identify the group $\{P\}$. For this purpose, suitable restrictions must be placed on the admissible GR-reference frames, i.e., coordinate systems, prescribed by means of Eqs.(1) and (2) which are realized by the following requirements:

- *LPT-requirement #1* - For the validity of GCP, the two space-times must coincide and be transformed in one another by means of LPT, i.e., $(\mathbf{Q}^4, g(r)) \equiv (\mathbf{Q}'^4, g'(r'))$, so that to define a single C^k -differentiable Lorentzian manifold with $k \geq 3$, i.e., have either signature $(+, -, -, -)$ or analogous permutations.
- *LPT-requirement #2* - These transformations must be assumed as purely local, so that in Eqs.(1) and (2) r'^μ and r^μ must depend only locally respectively on $r \equiv \{r^\mu\}$ and $r' \equiv \{r'^\mu\}$. In other words, the *local values* r^μ and r'^μ are required to be mutually mapped in each other by means of the same equations, with r'^μ (respectively r^μ) being a function of r^μ (and similarly r'^μ) *only*.
- *LPT-requirement #3* - The coordinates r^μ and r'^μ must realize *physical observables* and hence be prescribed in terms of real variables, while the functions relating them (P and P^{-1}) must be suitably smooth in the sense that they are of class $C^{(k)}$, with $k \geq 3$. This means that $(\mathbf{Q}^4, g(r))$ must realize a C^k -differentiable Lorentzian manifold with $k \geq 3$.
- *LPT-requirement #4* - Eqs.(1) and (2) generate the corresponding 4-vector transformation equations for the contravariant components of the displacement 4-vectors dr^μ and dr'^μ (see Eqs.(3)). Analogous transformation laws follow, of course, for the covariant components of the displacements, namely $dr_\nu = g_{\mu\nu}(r)dr^\mu$. In view of Eqs.(1) and (2), by construction \mathcal{J}_ν^μ and $(\mathcal{J}^{-1})_\nu^\mu$ are considered respectively local functions of $r' \equiv \{r'^\mu\}$ and $r \equiv \{r^\mu\}$ only and must necessarily coincide with the *gradient forms* (4)-(5). Nevertheless, since \mathcal{J}_ν^μ and $(\mathcal{J}^{-1})_\nu^\mu$ are mutually related being inverse matrices of each other and the point transformations are purely local, it follows that they can also both formally be regarded as functions respectively of the variables r' and r .
- *LPT-requirement #5* - In terms of the Jacobian matrix \mathcal{J}_ν^μ and its inverse $(\mathcal{J}^{-1})_\nu^\mu$ the fundamental *LPT 4-tensor transformation laws* for the group $\{P\}$ are set for arbitrary tensors. Consider, for example, the Riemann curvature tensor $R_{\sigma\mu\nu}^\rho(r)$. In terms of an arbitrary LPT it obeys the 4-tensor transformation law

$$R_{\sigma\mu\nu}^\rho(r) = \mathcal{J}_\sigma^\alpha (\mathcal{J}^{-1})_\alpha^\rho \mathcal{J}_\mu^k \mathcal{J}_\nu^m R_{\beta km}^{\prime\alpha}(r'). \quad (6)$$

The same transformation law also requires that 4-scalars must be left unchanged under the action of the group $\{P\}$. Thus, by construction the 4-scalar proper-time element ds , i.e., the Riemann distance defined in terms of the equation $ds^2 = g_{\mu\nu}(r)dr^\mu dr^\nu \equiv g^{\mu\nu}(r)dr_\mu dr_\nu$, must satisfy the transformation law

$$ds^2 = g_{\mu\nu}(r)dr^\mu dr^\nu = g'_{\mu\nu}(r')dr'^\mu dr'^\nu, \quad (7)$$

which can be equivalently expressed as

$$ds^2 = g^{\mu\nu}(r)dr_\mu dr_\nu = g'^{\mu\nu}(r')dr'_\mu dr'_\nu. \quad (8)$$

Furthermore, the covariant and contravariant components of the metric tensor, i.e., $g_{\mu\nu}(r)$ and $g^{\mu\nu}(r)$ and respectively $g'_{\mu\nu}(r')$ and $g'^{\mu\nu}(r')$, must satisfy respectively the *LPT 4-tensor transformation laws*

$$g'_{\mu\nu}(r') = \mathcal{J}_\mu^\alpha(r') \mathcal{J}_\nu^\beta(r') g_{\alpha\beta}(r), \quad (9)$$

$$g'^{\mu\nu}(r') = (\mathcal{J}^{-1}(r'))^\mu_\alpha (\mathcal{J}^{-1}(r'))^\nu_\beta g^{\alpha\beta}(r), \quad (10)$$

so that the validity of the scalar transformation laws (7) and (8) is warranted.

- *LPT-requirement #6* - Introducing the corresponding *Lagrangian form* of the same equations, obtained by parametrizing both r^μ and r'^μ in terms of suitably-smooth time-like world-lines $\{r^\mu(s), s \in I\}$ and $\{r'^\mu(s), s \in I\}$, Eqs.(1)-(2) take the equivalent form

$$\begin{cases} P : r^\mu(s) \rightarrow r'^\mu(s) = r'^\mu(r(s)), \\ P^{-1} : r'^\mu(s) \rightarrow r^\mu(s) = r^\mu(r'(s)), \end{cases} \quad (11)$$

whereby the displacement 4-vectors $dr^\mu \equiv dr^\mu(s)$ and $dr'^\mu \equiv dr'^\mu(s)$ can be viewed as occurring during the proper time ds . Then it follows that Eqs.(11) imply also suitable transformation laws for the 4-velocities $u^\mu(s) = dr^\mu(s)/ds$ and $u'^\mu(s) = dr'^\mu(s)/ds$, which by definition span the tangent space $T\mathbb{D}^4$. The latter are provided by the equations

$$\begin{cases} u^\mu(s) = \mathcal{J}_\nu^\mu(r') u'^\nu(s), \\ u'^\mu(s) = (\mathcal{J}^{-1})^\mu_\nu(r) u^\nu(s). \end{cases} \quad (12)$$

Notice that here also the Jacobian \mathcal{J}_ν^μ and its inverse $(\mathcal{J}^{-1})^\mu_\nu$ must be considered as s -dependent (but just only through $r' = r'(s)$ and $r = r(s)$ respectively), i.e., of the form

$$\mathcal{J}_\nu^\mu(r') = \mathcal{J}_\nu^\mu(r'(s)), \quad (13)$$

$$(\mathcal{J}^{-1})^\mu_\nu(r) = (\mathcal{J}^{-1})^\mu_\nu(r(s)). \quad (14)$$

- *LPT-requirement #7* - Finally, in terms of Eqs.(11) and (12) one notices that a LPT can be formally represented in terms of Lagrangian phase-space transformations of the type:

$$\begin{cases} \{r^\mu(s), u^\mu(s)\} \rightarrow \{r'^\mu(s), u'^\mu(s)\} = \{r'^\mu(r(s)), (\mathcal{J}^{-1})^\mu_\nu(r) u^\nu(s)\}, \\ \{r'^\mu(s), u'^\mu(s)\} \rightarrow \{r^\mu(s), u^\mu(s)\} = \{r^\mu(r'(s)), \mathcal{J}_\nu^\mu(r') u'^\nu(s)\} \end{cases} \quad (15)$$

(*LPT-phase-space transformation*), with the vectors $\{r^\mu(s), u^\mu(s)\}$ and $\{r'^\mu(s), u'^\mu(s)\}$ to be viewed as representing the phase-space states, endowed by 4-positions $r^\mu(s)$ and $r'^\mu(s)$ respectively, and corresponding 4-velocities $u^\mu(s)$ and $u'^\mu(s)$. Hence, by construction the transformation (15) warrants the scalar and tensor transformation laws (7) and (9) and preserves the structure of the space-time (\mathbf{Q}^4, g) .

This concludes the prescription of the LPT- functional setting required for the validity of GCP.

It must be stressed that its adoption is of paramount importance in the context of GR and in particular for the subsequent considerations regarding the physical interpretations of Einstein teleparallelism. This happens at least for the following three main motivations. The first one is that, in validity of the *LPT-requirements #1-#6*, and in particular the gradient-form requirement (4)-(5) for the Jacobian matrix, Eqs.(12) are equivalent to the Eulerian equations (1)-2) (and of course also to the corresponding Lagrangian equations (11)). Hence, *both equations actually allow one to identify uniquely the group $\{P\}$ (Proposition #1)*.

The second one concerns the very notion of particular solution to be adopted in the context of GR for the Einstein equation. In fact, if $g_{\mu\nu}(r)$ denotes a *parametrized-solution* of the same equation obtained with respect to a GR-frame r^μ , the notion of particular solution for the same equation is actually peculiar. Indeed, it must necessarily coincide with the whole equivalence class of parametrized-solutions, represented symbolically as $\{g_{\mu\nu}(r)\}$, which are mapped in each other by means of an arbitrary LPT of the group $\{P\}$. Such a property, which is actually a consequence of GCP (and consequently of Classical Tensor Analysis on Manifolds), is usually being referred to in GR as the so-called *principle of frame's (or observer's) independence* (*Proposition #2*).

The third motivation concerns the very notion of curved space-time $(\mathbf{Q}^4, g(r))$, compared to that of the Minkowski flat space-time (\mathbf{Q}^4, η) , which when expressed in orthogonal Cartesian coordinates $r'^\mu \equiv ((r'^0, (\mathbf{r}' \equiv x', y', z'))$ has the metric tensor $\eta_{\mu\nu} = \text{diag}\{1, -1, -1, -1\}$. A generic space-time of this type is characterized, by definition, by a

non-vanishing Riemann curvature 4-tensor $R_{\sigma\mu\nu}^\rho(r)$. As a consequence of the 4-tensor transformation laws (9)-(10) it follows that two generic space-times $(\mathbf{Q}^4, g(r))$ and $(\mathbf{Q}'^4, g'(r'))$ can be mapped in each other by means of LPTs, and hence actually coincide, only provided the respective metric tensors, and hence also the corresponding Riemann curvature 4-tensors, are transformed in each other via the same Eqs.(9)-(10). Hence, it is obvious that a generic curved space-time cannot be mapped into the said Minkowski space-time purely by means of a LPT (*Proposition #3*).

3 - EINSTEIN'S TELEPARALLEL GRAVITY AND THE TELEPARALLEL PROBLEM

Most of the historical developments achieved so far in GR since its original appearance in 1915 have been obtained in the framework of the GCP-setting of GR [8]. Nonetheless for a long time the issue has been debated whether Relativistic Classical Mechanics and Relativistic Classical theory of fields might exhibit in each case (possibly-different) non-local phenomena. In the literature there are several examples of studies aimed at extending in the context of GR the classical notions of local dynamics and local field interactions. A related question is, however, whether there actually exist additional non-local phenomena which might escape the validity of GCP and require the setup of a proper theoretical framework for their study.

As we intend to show, an instance of this type arises in the context of the so-called teleparallel approach to GR, also known as Einstein teleparallelism [9] (see also Refs.[16–18]). To state the issue in the appropriate physical context let us briefly highlight the basic ideas behind such an approach. This is based on the conjecture on Einstein part that at each point r^μ of the space-time manifold $(\mathbf{Q}^4, g(r))$ the corresponding tangent space $T\mathbb{D}^4$ can be “parallelized”. This means, in other words, that at all 4-positions $r^\mu \in (\mathbf{Q}^4, g(r))$ it should be possible to cast each tangent 4-vector $u^\mu(s)$ in the form

$$\begin{cases} u^\mu(s) = M_\nu^\mu u'^\nu(s), \\ u'^\mu(s) = (M^{-1})_\nu^\mu u^\nu(s), \end{cases} \quad (16)$$

with $\{M_\alpha^\mu\}$ being an invertible matrix with inverse $(M^{-1})_\mu^\alpha \equiv (M^{-1})_\mu^\alpha$. More precisely, according to Einstein’s approach the metric tensor of a generic curved space-time $(Q^4, g(r))$ should satisfy an equation in the form:

$$g_{\mu\nu}(r) = (M^{-1})_\mu^\alpha (M^{-1})_\nu^\beta \eta_{\alpha\beta}, \quad (17)$$

or equivalent

$$M_\alpha^\mu(r) M_\beta^\nu(r) g_{\mu\nu}(r) = \eta_{\alpha\beta}, \quad (18)$$

with $\eta_{\alpha\beta}$ being here the metric tensor associated with the flat Minkowski space-time $(\mathbf{Q}^4 \equiv \mathbf{M}^4, \eta)$ having the Lorentzian signature $(+, -, -, -)$. The goal is therefore to determine the map

$$\eta_{\alpha\beta} \leftrightarrow g_{\mu\nu}(r), \quad (19)$$

known as the *teleparallel transformation (TT)*, while Eq.(17) (or equivalent (18)) will be referred to as the *TT-problem*. For definiteness, it must be stressed here what appears to be the Einstein’s key assumption underlying these equations: it is understood in fact that in Eqs.(17) and (18) $\eta_{\alpha\beta}$ manifestly identifies the metric tensor of the Minkowski space-time (M^4, η) when expressed in terms of orthogonal Cartesian coordinates. On the other hand it is also understood that Eqs. (17) and (18) should include the identity transformation among their possible solutions. This means that for consistency $g_{\mu\nu}(r)$ can always be identified with the metric tensor of the curved space-time $(Q^4, g(r))$ when expressed as a local function of the *same Cartesian coordinates*. We shall return on this issue in Part 2. In the present paper such a viewpoint shall be consistently adopted in the subsequent considerations to be developed in Section 4.

The following additional remarks must also be made regarding the TT-problem.

- The first one concerns the interpretation of Eq.(18) in the so-called tetrad formalism. It implies, in fact, that for $\mu = 0, 3$ the fields $M_0^\mu(r)$, $M_1^\mu(r)$, $M_2^\mu(r)$ and $M_3^\mu(r)$ can simply be interpreted as a *tetrad basis*, i.e., a set of four independent real 4-vector fields that are mutually orthogonal, i.e., such that for $\alpha \neq \beta$:

$$e_\alpha^\mu(r) e_\beta^\nu(r) g_{\mu\nu}(r) = 0. \quad (20)$$

Also, all basis 4-vectors are unitary, in the sense that for all $\alpha = 0, 3$, $\left| M_\alpha^\mu(r) M_{(\alpha)}^\nu(r) g_{\mu\nu}(r) \right| = 1$, one of them ($M_0^\mu(r)$) being time-like and the others space-like, namely

$$\begin{aligned} M_0^\mu(r) M_0^\nu(r) g_{\mu\nu}(r) &= -1, \\ M_\alpha^\mu(r) M_{(\alpha)}^\nu(r) g_{\mu\nu}(r) &= 1, \end{aligned} \quad (21)$$

together span the 4-D tangent space at each point r^μ in the space-time (\mathbf{Q}^4, g) .

- The second remark is about the choice of the curved space-time $(\mathbf{Q}^4, g(r))$ in the TT-problem. It must be stressed, in fact, that the space-time $(\mathbf{Q}^4, g(r))$ should remain in principle arbitrary. Therefore, it should always be possible to identify $(\mathbf{Q}^4, g(r))$ with the curved space-time having signature different from that of the Minkowski space-time. Therefore, the solution of the TT-problem should be possible also in the case in which $(\mathbf{Q}^4, g(r))$ and (\mathbf{M}^4, η) have different signatures.
- The third remark is about the ultimate goal of Einstein teleparallelism. This emerges perspicuously from Eq.(17) (or equivalent its inverse represented by Eq.(18)). The determination of the matrix $M_\alpha^\mu(r)$ solution of such an equation will be referred to here as *TT- problem*. In fact, Eq.(17) - i.e., if a solution exists to such an equation - should permit one to relate curved and flat space-time metric tensors, respectively identified with $g_{\mu\nu}(r)$ and $\eta_{\alpha\beta}$.

From these premises it emerges, therefore, the fundamental problem of establishing a map between the generic curved space-time (\mathbf{Q}^4, g) indicated above and the Minkowski space-time (\mathbf{M}^4, η) , which should have a global validity, namely it should hold in the whole (\mathbf{Q}^4, g) or at least in a finite subset of the same space-time. However, such a kind of transformation cannot be realized by means of LPT of the type (1)-(2) in which $M_\alpha^\mu(r)$ is identified with the corresponding Jacobian (see Eq.(4) below). This happens because the teleparallel transformation cannot be realized by means of the group of LPT $\{P\}$ (see also the related *Proposition #3* indicated above). The issue arises whether in the context of GR the teleparallel transformation (17) (or equivalent its inverse, i.e., Eq.(18)) might actually still apply in the case of a more general type of *non-local point transformations* (NLPT), with the matrix $M_\alpha^\mu(r)$ to be identified with a corresponding suitably-prescribed Jacobian matrix.

The existence of such a class of generalized GR-reference frames and coordinate systems is actually suggested by the Einstein equivalence principle (EEP) itself. This is expressed by two separate propositions, which in the form presently known must both be ascribed to Albert Einstein's 1907 original formulation [34] (see also Ref.[35]). The part of EEP which is mostly relevant for the current discussion is the one usually referred to as the so-called weak equivalence principle (WEP). This is related, in fact, to the fundamental notion of equivalence between gravitational and inertial mass as well as to Albert Einstein's observation that the gravitational "force" as experienced locally while standing on a massive body is actually the same as the pseudo-force experienced by an observer in a non-inertial (accelerated) frame of reference. Apparently there is no unique formulation of WEP to be found in the literature. However, the form of WEP which is of key importance in the following consists in the two distinct claims by Einstein stating: *a)* the equivalence between accelerating frames and the occurrence of gravitational fields (see also Ref.[8]); *b)* that "*local effects of motion in a curved space (gravitation)*" should be considered as "*indistinguishable from those of an accelerated observer in flat space*" [34, 35]. Incidentally, it must be stressed that statement *b)* is the basis of Einstein's 1928 paper on teleparallelism.

From a historical perspective, the original introduction of WEP (and EEP) on the part of Albert Einstein was later instrumental for the development of GR. An interesting question concerns the conditions of validity of GCP and the choice of the class of LPTs for which WEP applies. In fact, based on the discussion above, the issue is whether it is possible to extend in such a framework the class of LPTs. In particular, here we intend to look for a more general group of point transformations, to be identified with NLPTs. These are distinguished from the class $\{P\}$ introduced above and form a group of transformations denoted here as *special NLPT-group* $\{P_S\}$. This new type of transformations connects two accelerating frames, namely curvilinear coordinate systems mutually related by means of suitable acceleration-dependent and necessarily non-local coordinate transformations. The latter should permit one to connect globally two suitable subsets of Lorentzian spaces which realize accessible domains (in the sense indicated below) and are endowed with different metric tensors having intrinsically-different Riemann tensors. Therefore, these transformations should have the property of being globally defined and, together with the corresponding inverse transformations, be respectively endowed with Jacobians $M_\alpha^\mu(r)$ and $(M^{-1}(r))^\mu_\nu$.

We intend to show that, provided suitable "ad hoc" restrictions are set on the class of manifolds among which NLPTs are going to be established, a non-trivial generalization of GR by means of the general NLPT-group $\{P_S\}$.

These will be shown to be realized in terms of a suitably-prescribed diffeomorphism between 4-dimensional Lorentzian space-times (\mathbf{Q}^4, g) and (\mathbf{Q}'^4, g') of the general form

$$P_g : r'^\mu \rightarrow r^\mu = r'^\mu \{r', [r', u']\}, \quad (22)$$

with inverse transformation

$$P_g^{-1} : r^\mu \rightarrow r'^\mu = r'^\mu \{r, [r, u]\}. \quad (23)$$

Here the squared brackets $[r', u']$ and $[r, u]$ denote possible suitable *non-local dependences* in terms of the 4-positions r'^μ , r^μ and corresponding 4-velocities $u^\mu \equiv \frac{dr^\mu}{ds}$ and $u'^\mu \equiv \frac{dr'^\mu}{ds}$ respectively. As a consequence, Eqs.(22)-(23) identify a new kind of point transformations, which unlike LPTs (see Eqs.(1) and (2)) are established between intrinsically different manifolds (\mathbf{Q}^4, g) and (\mathbf{Q}'^4, g') , i.e., which cannot be mapped in each other purely by means of LPTs.

4 - EXPLICIT SOLUTION OF THE TT-PROBLEM - THE NLPT-FUNCTIONAL SETTING

Let us now pose the problem of constructing explicitly the new type of point transformations, i.e. the NLPTs, which are involved in the representation problem of teleparallel gravity and identifying, in the process, the corresponding NLPT-functional setting.

For this purpose we introduce first the conjecture that, consistent with EEP, it should be possible to generate such a transformation introducing a suitable 4-velocity transformation $u^\mu \rightarrow u'^\mu$ which connects appropriate sets of GR-reference frames belonging to the two space-times indicated above. Indeed, it is physically conceivable the possibility of constructing “ad hoc” 4-velocity transformations which are not reducible to LPTs of the type (1) and (2). To show how this task can be achieved in practice, we notice that the transformation laws for the 4-velocity which are realized, by assumption, by Eqs.(16), necessarily imply the validity of corresponding transformation equations for the displacement 4-vectors $dr^\mu(s)$ and $dr'^\mu(s)$. These read manifestly

$$\begin{cases} dr^\mu(s) = M_\nu^\mu dr'^\nu(s), \\ dr'^\mu(s) = (M^{-1})_\nu^\mu dr^\nu(s), \end{cases} \quad (24)$$

where for generality M_ν^μ and $(M^{-1})_\nu^\mu$ are considered of the form $M_\nu^\mu = M_\nu^\mu(r', r)$ and $(M^{-1})_\nu^\mu = (M^{-1})_\nu^\mu(r, r')$. In analogy with Eqs.(13) and (14), when evaluated along the corresponding world-lines, it follows that they take the general functional form

$$M_\nu^\mu = M_\nu^\mu(r'(s), r(s)), \quad (25)$$

$$(M^{-1})_\nu^\mu = (M^{-1})_\nu^\mu(r(s), r'(s)), \quad (26)$$

with M_ν^μ and $(M^{-1})_\nu^\mu$ being now smooth functions of s through the variables $r(s) \equiv \{r^\mu(s)\}$ and $r'(s) \equiv \{r'^\mu(s)\}$. More precisely, in analogy to the LPT-requirements recalled above, the following prescriptions can be invoked to determine the NLPT-functional setting:

- *NLPT-requirement #1* - The coordinates r^μ and r'^μ realize by assumption physical observables and hence are prescribed in terms of real variables, while $(\mathbf{Q}^4, g(r))$ and (\mathbf{M}^4, η) must both realize C^k -differentiable Lorentzian manifolds, with $k \geq 3$.
- *NLPT-requirement #2* - The matrices M_ν^μ and $(M^{-1})_\nu^\mu$ are assumed to be *locally* smoothly-dependent only on 4-position, while admitting at the same time also possible non-local dependences. More precisely, in the case of the Jacobian $M_\nu^\mu(r', r)$ the second variable $r \equiv \{r^\mu\}$ which enters the same function can contain in general both local and non-local implicit dependences, the former ones in terms of r'^μ . Similar considerations apply to the inverse matrix $(M^{-1})_\nu^\mu(r, r')$, which besides local explicit and implicit dependences in terms of r^μ , may generally include additional non-local dependences through the variable $r' \equiv \{r'^\mu\}$.
- *NLPT-requirement #3* - The Jacobian matrix M_ν^μ and its inverse $(M^{-1})_\nu^\mu$ are assumed to be generally non-gradient. In other words, at least in a subset of the two space times $(\mathbf{M}^4, \eta) \equiv (\mathbf{Q}'^4, g')$ and (\mathbf{Q}^4, g) :

$$M_\nu^\mu(r', r) \neq \frac{\partial r^\mu(r', r)}{\partial r'^\nu}, \quad (27)$$

$$(M^{-1})_\nu^\mu(r, r') \neq \frac{\partial r'^\mu(r, r')}{\partial r^\nu}, \quad (28)$$

while elsewhere they can still recover the gradient form (4) and (5), namely

$$M_{\nu}^{\mu}(r', r) = \frac{\partial r^{\mu}(r', r)}{\partial r'^{\nu}}, \quad (29)$$

$$(M^{-1})_{\nu}^{\mu}(r, r') = \frac{\partial r'^{\mu}(r, r')}{\partial r^{\nu}}. \quad (30)$$

In both cases the partial derivative are performed with respect to the local dependences only.

- *NLPT-requirement #4* - Introducing the (proper-time) line elements ds, ds' in the two space-times (\mathbf{Q}^4, g) and $(\mathbf{M}^4, \eta) \equiv (\mathbf{Q}^4, g')$ defined respectively according to Eq.(8) and so that

$$ds^2 = g_{\mu\nu}(r) dr^{\mu} dr^{\nu}, \quad (31)$$

$$ds'^2 = g'_{\mu\nu}(r') dr'^{\mu} dr'^{\nu} \equiv \eta_{\mu\nu} dr'^{\mu} dr'^{\nu}, \quad (32)$$

the *isometric constraint condition*

$$ds = ds' \quad (33)$$

is set. This implies that the equation

$$g_{\mu\nu}(r) dr^{\mu} dr^{\nu} = \eta_{\mu\nu} dr'^{\mu} dr'^{\nu} \quad (34)$$

must hold.

- *NLPT-requirement #5* - Finally, we shall assume that the 4-positions $r^{\mu}(s)$ and $r'^{\mu}(s)$ spanning the corresponding space-times (\mathbf{Q}^4, g) and (\mathbf{M}^4, η) are represented in terms of the same Cartesian coordinates, i.e.,

$$r^{\mu} \equiv \{ct, (\mathbf{r} \equiv x, y, z)\} \quad (35)$$

and

$$r'^{\mu} \equiv \{ct', (\mathbf{r}' \equiv x', y', z')\}. \quad (36)$$

Let us now briefly analyze the implications of these Requirements. First, Eqs.(24) (or equivalent Eqs.(16)) can be integrated at once performing the integration along suitably-smooth time- (or space-) like world lines $r^{\mu}(s)$ and $r'^{\mu}(s)$

$$\begin{cases} P_S : r^{\mu}(s) = r^{\mu}(s_o) + \int_{s_o}^s d\bar{s} M_{\nu}^{\mu}(r', r) u'^{\nu}(\bar{s}), \\ P_S^{-1} : r'^{\mu}(s) = r'^{\mu}(s_o) + \int_{s_o}^s d\bar{s} (M^{-1})_{\nu}^{\mu}(r, r') u^{\nu}(\bar{s}), \end{cases} \quad (37)$$

where the initial condition is set

$$r^{\mu}(s_o) = r'^{\mu}(s_o). \quad (38)$$

Transformations (37) will be referred to as *special NLPT in Lagrangian form*, the family of such transformations identifying the special NLPT-group $\{P_S\}$, i.e., a suitable subset of the group of general NLPT-group $\{P_g\}$. The subsets of two space-times (\mathbf{Q}^4, g) and $(\mathbf{Q}^4, g') \equiv (\mathbf{M}^4, \eta)$ which are mapped in each other by a special NLPT, both assumed to have non-vanishing measure, will be referred to as *accessible sub-domains*. Depending on the signature of (\mathbf{Q}^4, g) an accessible subset of the same space-time can be covered in principle either by time- (or space-) like world-lines.

Notice that the Jacobians $M_{\nu}^{\mu}(r', r)$ and $(M^{-1})_{\nu}^{\mu}(r, r')$ remain still in principle arbitrary. In particular, in case they take the gradient forms (29) and (30) the Lagrangian LPT defined by Eqs.(11) is manifestly recovered. Furthermore, Eqs.(16), or equivalent Eqs.(37), can be also represented in terms of the equations for the infinitesimal 4-displacements, given by Eq.(24). In particular, assuming the matrix M_{ν}^{μ} to be continuously connected to the identity δ_{ν}^{μ} , implies that the Jacobian matrix M_{ν}^{μ} and its inverse $(M^{-1})_{\nu}^{\mu}$ can always be represented in the form

$$M_{\nu}^{\mu} = \delta_{\nu}^{\mu} + \mathcal{A}_{\nu}^{\mu}(r, r'), \quad (39)$$

$$(M^{-1})_{\nu}^{\mu} = \delta_{\nu}^{\mu} + \mathcal{B}_{\nu}^{\mu}(r, r'), \quad (40)$$

with \mathcal{A}_ν^μ and \mathcal{B}_ν^μ being suitable *transformation matrices*, which are mutually related by matrix inversion. Hence, in terms of Eqs.(39)-(40), the special NLPT in Lagrangian form (37) yields then the corresponding Lagrangian and Eulerian forms:

$$\begin{cases} r^\mu(s) = r'^\mu(s) + \int_{s_o}^s d\bar{s} \mathcal{A}_\nu^\mu(r', r) u'^\nu(\bar{s}) \\ r'^\mu(s) = r^\mu(s) + \int_{s_o}^s d\bar{s} \mathcal{B}_\nu^\mu(r', r) u^\nu(\bar{s}) \end{cases} \quad (41)$$

$$\begin{cases} r^\mu = r'^\mu + \int_{r'^\nu(s_o)}^{r'^\nu} dr'^\nu \mathcal{A}_\nu^\mu(r', r) \\ r'^\mu = r^\mu + \int_{r^\nu(s_o)}^{r^\nu} dr^\nu \mathcal{B}_\nu^\mu(r, r') \end{cases} \quad (42)$$

We stress that, in difference with the treatment of LPT, in the proper-time integral on the rhs of Eqs.(37) and (41) the tangent-space curve $u'^\nu(\bar{s})$ (respectively $u^\nu(\bar{s})$) must be considered as an independent variable. This is a peculiar feature of Eqs.(37) which cannot be avoided. The reason lies in the fact that there is no way by which $u'^\nu(\bar{s})$ (and $u^\nu(\bar{s})$) can be uniquely prescribed by means of the same equations. Indeed, equations (37) (or equivalent (41) and (42)) together with Eqs.(16) truly establish a phase-space transformation of the form:

$$\begin{cases} \{r^\mu(s), u^\mu(s)\} \rightarrow \{r'^\mu(s), u'^\mu(s)\} = \{r'^\mu \{r(s), [r, u]\}, (\mathcal{M}^{-1})_\nu^\mu(r) u^\nu(s)\} \\ \{r'^\mu(s), u'^\mu(s)\} \rightarrow \{r^\mu(s), u^\mu(s)\} = \{r^\mu \{r'(s), [r', u']\}, \mathcal{M}_\nu^\mu(r') u'^\nu(s)\} \end{cases} \quad (43)$$

This will be referred to as *NLPT-phase-space transformation*. The latter apply to a new type of reference frame, denoted as *extended GR-frames*, which are represented by the vectors $\{r^\mu(s), u^\mu(s)\}$ and $\{r'^\mu(s), u'^\mu(s)\}$ respectively. These can be viewed as phase-space states (of the corresponding extended GR-frames) having respectively 4-positions $r^\mu(s)$ and $r'^\mu(s)$ and 4-velocities $u^\mu(s)$ and $u'^\mu(s)$. Finally, let us mention that the transformation (43), in contrast with (15), obviously does not preserve the structure of the space-times (\mathbf{Q}^4, g) and (\mathbf{M}, η) . Nevertheless the scalar transformation law (7) is still by construction warranted, while at the same time the metric tensor satisfies by construction the TT-problem, i.e., Eq.(17).

Let us now show how the matrices \mathcal{A}_ν^μ and \mathcal{B}_ν^μ can be explicitly determined in terms of the teleparallel transformation (17). The relevant results, which actually prescribe the general form of related NLPT, are summarized by the following proposition.

THM.1 - Realization of the special NLPT-group $\{P_S\}$ for the TT-problem

Let us assume that (\mathbf{Q}^4, g) and $(\mathbf{Q}'^4, g') \equiv (\mathbf{M}'^4, \eta)$ identify respectively a generic curved space-time and the Minkowski space-time both parametrized in terms of orthogonal Cartesian coordinates (47) and (36).

Then, given validity of the NLPT-Requirement #1-#5, the following propositions hold.

P₁) In the accessible sub-domain of (\mathbf{Q}^4, g) the teleparallel transformation (17) (or equivalent its inverse, i.e., Eq.(18)), relating (\mathbf{Q}^4, g) with the Minkowski space-time $(\mathbf{Q}'^4, g') \equiv (\mathbf{M}'^4, \eta)$, is realized by a non-local point transformation of the type (37), or equivalent (41) and (42), with a Jacobian M_μ^ν and its inverse $(M^{-1})_\mu^\alpha$ being of the form (25) and (26) respectively. This is required to satisfy the 4-tensor transformation law prescribed by the matrix equations

$$g_{\mu\nu}(r) = (M^{-1})_\mu^\alpha(r, r') (M^{-1})_\nu^\beta(r, r') \eta_{\alpha\beta}, \quad (44)$$

and similarly its inverse (see Eq.(18)) where $g_{\mu\nu}(r)$ identifies a prescribed symmetric metric tensor associated with the space-time (\mathbf{Q}^4, g) , by assumption expressed in the Cartesian coordinates (35). Hence, $(M^{-1})_\mu^\alpha(r, r')$ necessarily coincides with the Jacobian matrix of the TT-problem (see Eq.17)).

P₂) The set of special NLPT has the structure of a group.

Proof - Let us prove proposition P₁. For this purpose it is sufficient to construct explicitly a possible, i.e., non-unique, realization of the NLPT and the corresponding set $\{P_S\}$, satisfying Eq.(44). In fact, let us consider the equation for the infinitesimal 4-displacement dr'^μ (see Eq.(24)), which in validity of Eq.(40) becomes

$$dr'^\mu = [\delta_\nu^\mu + \mathcal{B}_\nu^\mu(r, r')] dr^\nu, \quad (45)$$

and similarly

$$dr^\mu = [\delta_\nu^\mu + \mathcal{A}_\nu^\mu(r, r')] dr'^\nu, \quad (46)$$

where the matrices $\mathcal{B}_\nu^\mu(r, r')$ and $\mathcal{A}_\nu^\mu(r, r')$ are suitably related. Substituting dr'^μ on the rhs of the last equation and invoking the independence of the components of the infinitesimal displacement dr^μ , this means for consistency that the covariant components of the metric tensor, i.e., $g_{\mu\nu}(r)$ and respectively $g'_{\mu\nu}(r') \equiv \eta_{\mu\nu}$ must satisfy the 4-tensor transformation law (44). Such a tensor equation delivers, therefore, a set of 10 algebraic equations. Their solution can be determined in a straightforward way for the 16 components of the matrix $\mathcal{B}_\nu^\mu(r, r')$. For example, one of these equations reads

$$g_{00}(r) = [1 + \mathcal{B}_0^0(r, r')]^2 - [\mathcal{B}_0^1(r, r')]^2 - [\mathcal{B}_0^2(r, r')]^2 - [\mathcal{B}_0^3(r, r')]^2. \quad (47)$$

The remaining equations following from Eq.(44) are not reported here for brevity.

One can nevertheless show that the solution to this set is non-unique. In fact, due to the freedom in the choice of the matrix elements of $\mathcal{B}_\nu^\mu(r, r')$, the latter can in principle be chosen arbitrarily by suitably prescribing 6 components of the same matrix. A particular solution is obtained, for example, by requiring validity of the constraint equations

$$\begin{aligned} \mathcal{B}_3^0(r, r') &= \mathcal{B}_3^1(r, r') = \mathcal{B}_0^1(r, r') = 0, \\ \mathcal{B}_0^2(r, r') &= \mathcal{B}_1^2(r, r') = \mathcal{B}_3^2(r, r') = 0. \end{aligned} \quad (48)$$

The surviving components of \mathcal{B}_ν^μ are then determined by the same algebraic equations of the set (44). From these considerations it follows that necessarily it must be $\mathcal{B}_\nu^\mu = \mathcal{B}_\nu^\mu(r)$. In particular, here we notice that all diagonal components $\mathcal{B}_i^i(r)$ for $i = 0, 3$ can be viewed as determined, up to an arbitrary sign, by the diagonal components of the metric tensor $g_{\mu\mu}(r)$. Instead, the remaining non-diagonal matrix elements are then prescribed in terms of the non-diagonal components of the metric tensor, which follow analogously from the corresponding 6 equations of the set. Then, both the 4-displacement transformations (45) and their inverse (46) ones exist and can be non-uniquely prescribed. An example of possible realization is given by

$$\begin{cases} dr'^0 = [1 + \mathcal{B}_0^0] dr^0 + \mathcal{B}_2^0 dr^2 \\ dr'^1 = [1 + \mathcal{B}_1^1] dr^1 + \mathcal{B}_0^1 dr^0 + \mathcal{B}_2^1 dr^2 \\ dr'^2 = [1 + \mathcal{B}_2^2] dr^2 \\ dr'^3 = [1 + \mathcal{B}_3^3] dr^3 + \mathcal{B}_2^3 dr^2 \end{cases}, \quad (49)$$

with determinant

$$\begin{vmatrix} 1 + \mathcal{B}_0^0 & 0 & \mathcal{B}_2^0 & 0 \\ \mathcal{B}_0^1 & 1 + \mathcal{B}_1^1 & \mathcal{B}_2^1 & 0 \\ 0 & 0 & [1 + \mathcal{B}_2^2] & 0 \\ 0 & 0 & \mathcal{B}_2^3 & [1 + \mathcal{B}_3^3] \end{vmatrix} = \prod_{i=0,3} (1 + \mathcal{B}_i^i), \quad (50)$$

to be assumed as non-vanishing, and with inverse transformation

$$\begin{cases} dr^0 = \frac{1}{1 + \mathcal{B}_0^0} \left[dr'^0 + \frac{\mathcal{B}_2^0}{1 + \mathcal{B}_2^2} dr'^2 \right] \\ dr^1 = \frac{1}{1 + \mathcal{B}_1^1} \left[dr'^1 - \frac{\mathcal{B}_0^1}{1 + \mathcal{B}_0^0} dr'^0 - \frac{1}{1 + \mathcal{B}_2^2} \frac{\mathcal{B}_0^1 \mathcal{B}_2^0}{1 + \mathcal{B}_0^0} dr'^2 \right] \\ dr^2 = \frac{1}{1 + \mathcal{B}_2^2} dr'^2 \\ dr^3 = \frac{1}{1 + \mathcal{B}_3^3} \left[dr'^3 - \frac{\mathcal{B}_2^3}{1 + \mathcal{B}_2^2} dr'^2 \right] \end{cases}. \quad (51)$$

In particular, from Eqs.(51) one can easily evaluate in terms of $\mathcal{B}_\nu^\mu(r)$ the precise expression taken by of the matrix \mathcal{A}_ν^μ which appears in Eqs.(58). Hence one finds that necessarily $\mathcal{A}_\nu^\mu = \mathcal{A}_\nu^\mu(r)$, with $r \equiv \{r^\mu\}$ being now considered as prescribed by means of the NLPTs (41). Finally, the corresponding finite NLPTs generated by Eqs.(58) and (51) can always be equivalently represented in terms Eqs.(37).

Next, let us prove proposition P₂. For this purpose we first notice that the Jacobian $\mathcal{J}_\nu^\mu \equiv \delta_\nu^\mu + \mathcal{A}_\nu^\mu(r, r')$ admits the inverse which by construction coincides with $(\mathcal{J}^{-1})_\rho^\mu \equiv \delta_\nu^\mu + \mathcal{B}_\nu^\mu(r', r)$. Furthermore, let us consider two special NLPTs

$$\mathcal{J}_{(i)\nu}^\mu \equiv \delta_\nu^\mu + \mathcal{A}_{(i)\nu}^\mu(r_{(i)}, r') \quad (52)$$

which map the space-times $(\mathbf{Q}_{(i)}^4, g)$ (for $i = 1, 2$) onto (\mathbf{M}^4, η) . Requiring that both the corresponding admissible subsets of (\mathbf{M}^4, η) and their intersection have a non-vanishing measure, the product of two special NLPT is defined on such a set. Its Jacobian is

$$\mathcal{J}_\nu^\mu = \left(\delta_\alpha^\mu + \mathcal{A}_{(1)\alpha}^\mu(r_{(1)}, r') \right) \left(\delta_\nu^\alpha + \mathcal{B}_{(2)\nu}^\alpha(r', r_{(2)}) \right) = \left(\delta_\nu^\mu + \mathcal{C}_\nu^\mu(r_{(1)}, r', r_{(2)}) \right), \quad (53)$$

with $C_\nu^\mu(r_{(1)}, r', r_{(2)}) \equiv \mathcal{A}_{(1)\nu}^\mu(r_{(1)}, r') + \mathcal{B}_{(2)\nu}^\mu(r', r_{(2)}) + \mathcal{A}_{(1)\alpha}^\mu(r_{(1)}, r')\mathcal{B}_{(2)\nu}^\alpha(r', r_{(2)})$. It follows that in such a circumstance the product of the two special NLPT belongs necessarily to the same set $\{P_S\}$, which is therefore a group.

Q.E.D.

THM.1 provides the formal solution of the Einstein's TT-problem in the framework of the theory of NLPT. This is achieved by means of the introduction of a non-local phase-space transformation of the type (15), which is realized by means of a special NLPT (37) and the corresponding 4-velocity transformation law (16). In this reference the following comments must be mentioned.

- First, the NLPT-functional setting has been prescribed in terms of the special NLPT-group $\{P_S\}$, determined here by Eqs.(37) together with the *NLPT-Requirements #1-#4*.
- Due to the non-uniqueness of the matrix $\mathcal{B}_\nu^\mu(r)$ solution of the TT- problem (see Eq.(44)), and of the related matrix \mathcal{A}_ν^μ , the realization of the NLPT transformation (58) [and hence (51)] yielding the solution of the TT-problem is manifestly non-unique too. For a prescribed curved space-time (\mathbf{Q}^4, g) which is parametrized in terms of the Cartesian coordinates, the ensemble of NLPTs which provide particular solutions of the TT-problem will be denoted as $\{P_g\}_{TT}$.
- Both for Eqs.49) and (51) the corresponding Jacobians determined by means of Eqs.(39) and (40) take by construction, and consistent with Eqs.(27)-(28), a manifest *non-gradient form*. This follows immediately from *Proposition #1* thanks to the validity of Eq.(44) and the requirement that (\mathbf{Q}^4, g) is a curved space-time.
- In terms of the Jacobian matrix $M_\nu^\mu(r', r)$ (and its inverse $(M^{-1})_\nu^\mu(r, r')$) Eq.(44) means that $g_{\mu\nu}(r)$ should actually satisfy the original Einstein's equations (17) and (18). The latter can be interpreted as *4-tensor transformation laws* for the matrix tensor $g_{\mu\nu}(r)$.
- Similarly and in analogy with Eq. (7) holding in the case of LPT, the validity of the scalar transformation law (8) is warranted also in the case of NLPT, thanks to the transformation law (44).
- Finally, the transformation law (44) for the metric tensor can be interpreted as tensor transformation law with respect to the special NLPT-group $\{P_S\}$. This will be referred to as *NLPL 4-tensor transformation law*. In terms of the same Jacobian matrix $M_\nu^\mu(r', r)$ and its inverse $(M^{-1})_\nu^\mu(r, r')$, analogous NLPT 4-tensor transformation laws can be set in principle for tensors of arbitrary order. Nevertheless, it must be noted that - specifically because of the validity of the same transformation law (44) - such a type of tensor transformation laws cannot be fulfilled by the Riemann curvature tensor $R_{\sigma\mu\nu}^\rho(r)$, the reason being that it manifestly vanishes identically in the case of the Minkowski space-time.

5 - CONDITIONS OF EXISTENCE OF SPECIAL NLPT FOR THE TT-PROBLEM

A fundamental aspect of the theory developed here concerns the conditions of existence of the family of special NLPT determined according to THM.1. In this regard we notice that the identification of the physical domain of existence involves the (possibly non-unique) prescription of the actual possible realization of the NLPT and of the corresponding subset of (\mathbf{Q}^4, g) which can be mapped onto the Minkowski space-time $(\mathbf{Q}'^4, g') \equiv (\mathbf{M}'^4, \eta)$. It is obvious that NLPT, just like LPT, can only be defined in the accessible sub-domains of (\mathbf{Q}^4, g) , namely the connected subsets which in the curved space-time can be covered by time- (or space-) like world-lines $r^\mu(s)$ which are endowed with a finite 4-velocity. Nevertheless, the components of the same 4-velocity can still be in principle arbitrarily-large, so that the corresponding world-line can be arbitrarily close to light trajectories (and therefore to the light cones).

Another aspect of the existence problem for NLPTs is related to the solubility conditions of the algebraic equations arising in THM.1, which follow from the requirement that all components of the matrix $\mathcal{B}_\nu^\mu(r, r')$ should be real. For example, in the case of Eq.(47) the corresponding condition is determined by the inequality

$$g_{00}(r) + [\mathcal{B}_0^1(r, r')]^2 + [\mathcal{B}_0^2(r, r')]^2 + [\mathcal{B}_0^3(r, r')]^2 \geq 0 \quad (54)$$

It must be stressed that the validity of inequalities of this type for the remaining equations in general cannot be warranted in the whole admissible subset of the space-time (\mathbf{Q}^4, g) , i.e., in particular in the subset in which $ds^2 > 0$. On the other hand, "*a priori*" the symmetric metric tensor $g_{\mu\nu}(r)$ must be regarded in principle as completely

arbitrary. Hence it is obvious that such inequalities following from THM.1 cannot place any “unreasonable” physical constraint on the same tensor $g_{\mu\nu}(r)$.

In fact, consider the case in which the metric tensor $g_{\mu\nu}(r)$ has the signature $(+, -, -, -)$ and is also diagonal, namely $g_{\mu\nu}(r) = \text{diag}\{g_{00}(r), g_{11}(r), g_{22}(r), g_{33}(r)\}$. Then, necessarily the metric tensor must be such that everywhere in the same admissible subset $g_{00}(r) > 0$, while $g_{11}(r), g_{22}(r), g_{33}(r) < 0$. As a consequence the functional class $\{P_g\}_{TT}$ contains transformations which may not exist everywhere in the same set. In fact, some of the inequalities of the group (54) which involve the spatial components, i.e., $g_{ii}(r)$ (with $i = 1, 2, 3$), must be considered as local, i.e., are subject to the condition of local validity of the same inequalities. Although NLPT of this kind are physically admissible, the question arises whether particular solutions actually exist which are not required to fulfill the same inequalities (54). These solutions, if they actually exist, have therefore necessarily a global character, i.e., they are defined everywhere in the same admissible subset of (\mathbf{Q}^4, g) . In view of these considerations, since the only acceptable physical restriction on $g_{\mu\nu}(r)$ concerns its signature, it can be shown that global validity is warranted everywhere in (\mathbf{Q}^4, g) provided the following two sets of constraints are required to hold:

$$[\mathcal{B}_0^1(r, r')]^2 + [\mathcal{B}_0^2(r, r')]^2 + [\mathcal{B}_0^3(r, r')]^2 = 0, \quad (55)$$

and

$$\begin{aligned} [\mathcal{B}_1^0(r, r')]^2 - [\mathcal{B}_1^2(r, r')]^2 - [\mathcal{B}_1^3(r, r')]^2 &\geq -\inf\{g_{11}(r)\}, \\ [\mathcal{B}_2^0(r, r')]^2 - [\mathcal{B}_2^1(r, r')]^2 - [\mathcal{B}_2^3(r, r')]^2 &\geq -\inf\{g_{22}(r)\}, \\ [\mathcal{B}_3^0(r, r')]^2 - [\mathcal{B}_3^1(r, r')]^2 - [\mathcal{B}_3^2(r, r')]^2 &\geq -\inf\{g_{33}(r)\}. \end{aligned} \quad (56)$$

The first equations actually requires the following 3 independent equations

$$\mathcal{B}_0^1(r, r') = \mathcal{B}_0^2(r, r') = \mathcal{B}_0^3(r, r') = 0 \quad (57)$$

to apply separately. Particular solutions of the components of \mathcal{B}_ν^μ satisfying the 3 constraint equations (57) and either the 3 inequalities (56), or corresponding equations obtained replacing the inequality symbol with $=$, will be denoted respectively as *partially unconditional* or *unconditional*. In both cases it is immediate to show that these solutions are non-unique, even if in all cases the transformation matrix is again a local function of r , i.e., $\mathcal{B}_\nu^\mu = \mathcal{B}_\nu^\mu(r)$.

In particular, here we notice that all the diagonal components $\mathcal{B}_i^i(r)$ for $i = 0, 1, 3$ can be viewed as determined, up to an arbitrary sign, by the diagonal components of the metric tensor $g_{\mu\mu}(r)$. Instead, the remaining non-diagonal matrix elements are then prescribed in terms of the non-diagonal components of the metric tensor, which follow analogously from the set of equations mentioned in THM.1. In validity of the constraints given above, i.e., both for partially unconditional or unconditional particular solutions, the 4-displacement transformations (45) become

$$\left\{ \begin{aligned} dr'^0 &= [1 + \mathcal{B}_0^0] dr^0 + \mathcal{B}_1^0 dr^1 + \mathcal{B}_2^0 dr^2 + \mathcal{B}_3^0 dr^3 \\ dr'^1 &= [1 + \mathcal{B}_1^1] dr^1 + \mathcal{B}_2^1 dr^2 + \mathcal{B}_3^1 dr^3 \\ dr'^2 &= [1 + \mathcal{B}_2^2] dr^2 + \mathcal{B}_1^2 dr^1 + \mathcal{B}_3^2 dr^3 \\ dr'^3 &= [1 + \mathcal{B}_3^3] dr^3 + \mathcal{B}_1^3 dr^1 + \mathcal{B}_2^3 dr^2 \end{aligned} \right. \quad (58)$$

Similarly, one can show that also the corresponding inverse NLPT exist.

Another relevant issue concerns the role of the non-local dependences entering the integral equations (41) and (42), which determine the general form of the NLPT. We notice, in fact, that Eqs.(16) although formally analogous to the 4-velocity transformation laws generated by LPT, i.e., Eqs.(12), are actually peculiar. This feature is reflected also in the transformation matrices \mathcal{A}_ν^μ and \mathcal{B}_ν^μ . In fact, although in view of the discussion given above both $\mathcal{B}_\nu^\mu(r)$ and $\mathcal{A}_\nu^\mu(r)$ are simply local functions of $r \equiv \{r^\mu\}$, it is obvious that the transformations Eqs.(37) have manifestly a non-local character.

6 - A SAMPLE CASE: SOLUTION OF THE TT-PROBLEM FOR DIAGONAL METRIC TENSORS

As pointed out above the theory of special NLPT must in principle hold also when the space-times (\mathbf{Q}^4, g) and $(\mathbf{Q}'^4, g') \equiv (\mathbf{M}'^4, \eta)$ have different signatures. In particular, if (\mathbf{Q}^4, g) coincides with a flat space-time, then it might still have in principle an arbitrary signature. To clarify this important point we present in this section a sample application. For definiteness, let us consider here a curved space-time (\mathbf{Q}^4, g) which is diagonal when expressed in terms of Cartesian coordinate. The following two possible realizations are considered

- A) $\text{diag}(g_{\mu\nu}) \equiv \text{diag}(S_0(r), -S_1(r), -S_2(r), -S_3(r))$.
- B) $\text{diag}(g_{\mu\nu}) \equiv \text{diag}(-S_0(r), S_1(r), -S_2(r), -S_3(r))$.

In both cases here the functions $S_\mu(r)$ are assumed to be *prescribed* real functions which are strictly positive for all $r \equiv r^\mu \in (\mathbf{Q}^4, g)$. Since by construction the Riemannian distance ds is left invariant by arbitrary NLPTs, it follows that in the two cases either the differential identity

$$\begin{aligned} ds^2 &= S_0 (dr^0)^2 - S_1 (dr^1)^2 - S_2 (dr^2)^2 - S_3 (dr^3)^2 \\ &= (dr'^0)^2 - (dr'^1)^2 - (dr'^2)^2 - (dr'^3)^2 \end{aligned} \quad (59)$$

or

$$\begin{aligned} ds^2 &= S_0 (dr^0)^2 - S_1 (dr^1)^2 - S_2 (dr^2)^2 - S_3 (dr^3)^2 \\ &= -(dr'^0)^2 + (dr'^1)^2 - (dr'^2)^2 - (dr'^3)^2 \end{aligned} \quad (60)$$

respectively must hold. Let us point out the solutions of the TT-problem, i.e., Eq.(17) or equivalent (18), in the two cases.

Solution of case A)

In validity of Eq.(59), if one adopts a special NLPT of the form

$$dr^\mu = \left(1 + A_{(\mu)}^{(\mu)}(r', r)\right) dr'^{(\mu)}, \quad (61)$$

in terms of Eq.(18) this delivers for diagonal matrix elements $A_{(\mu)}^\mu(r', r)$ for all $\mu = 0, 3$ the equations

$$1 = S_\mu(r) \left(1 + A_{(\mu)}^{(\mu)}(r', r)\right)^2, \quad (62)$$

with the formal solutions

$$A_{(\mu)}^\mu(r', r) = \sqrt{\frac{1}{S_{(\mu)}(r)}} - 1. \quad (63)$$

Notice that here only the positive algebraic roots have been retained in order to recover from Eq.(63) the identity transformation when letting $S_\mu(r) = 1$. From Eq.(41) one obtains therefore the special NLPT

$$r^\mu(s) = r^\mu(s_o) + \int_{s_o}^s ds \frac{dr'^{(\mu)}(s)}{ds} \sqrt{\frac{1}{S_{(\mu)}(r)}}, \quad (64)$$

where in the integrand r is to be considered as an implicit function of r' and, as indicated above, $\frac{dr'^{(\mu)}(s)}{ds}$ remains still arbitrary. Thus, explicit solution of Eq.(64) can be obtained by suitably prescribing $\frac{dr'^{(\mu)}(s)}{ds}$.

Solution of case B)

Let us now consider the solution of the TT-problem when Eq.(60) applies. For definiteness, let us look for a special NLPT of the type:

$$dr^0 = M_{(1)}^{(0)}(r', r) dr'^{(1)}, \quad (65)$$

$$dr^1 = M_{(0)}^{(1)}(r', r) dr'^{(0)}, \quad (66)$$

$$dr^2 = M_{(2)}^{(2)}(r', r) dr'^{(2)}, \quad (67)$$

$$dr^3 = M_{(3)}^{(3)}(r', r) dr'^{(3)}. \quad (68)$$

In terms of Eq.(18) this delivers for diagonal matrix elements $M_{(\mu)}^{(\mu)}$ the equations

$$1 = S_1(r) M_{(0)}^{(1)}(r', r)^2, \quad (69)$$

$$1 = S_0(r) M_{(1)}^{(0)}(r', r)^2, \quad (70)$$

$$1 = S_2(r) M_{(2)}^{(2)}(r', r)^2, \quad (71)$$

$$1 = S_3(r) M_{(3)}^{(3)}(r', r)^2, \quad (72)$$

with the formal solutions

$$M_{(0)}^{(1)}(r', r) = \sqrt{\frac{1}{S_{(1)}(r)}}, \quad (73)$$

$$M_{(1)}^{(0)}(r', r)^2 = \sqrt{\frac{1}{S_{(0)}(r)}}, \quad (74)$$

$$M_{(2)}^{(2)}(r', r)^2 = \sqrt{\frac{1}{S_{(2)}(r)}}, \quad (75)$$

$$M_{(3)}^{(3)}(r', r)^2 = \sqrt{\frac{1}{S_{(3)}(r)}}. \quad (76)$$

Hence, the corresponding NLPT in integral form are found to be in this case:

$$r^0(s) = r^0(s_o) + \int_{s_o}^s ds \frac{dr'^{(1)}}{ds} \sqrt{\frac{1}{S_{(0)}(r)}}, \quad (77)$$

$$r^1(s) = r^1(s_o) + \int_{s_o}^s ds \frac{dr'^{(0)}}{ds} \sqrt{\frac{1}{S_{(1)}(r)}}, \quad (78)$$

$$r^2(s) = r^2(s_o) + \int_{s_o}^s ds \frac{dr'^{(2)}}{ds} \sqrt{\frac{1}{S_{(2)}(r)}}, \quad (79)$$

$$r^2(s) = r^2(s_o) + \int_{s_o}^s ds \frac{dr'^{(3)}}{ds} \sqrt{\frac{1}{S_{(2)}(r)}}, \quad (80)$$

where, again, in the integrands r is to be considered as an implicit function of r' while $\frac{dr'^{(i)}}{ds}$ has to be suitably prescribed.

Cases A and B correspond respectively to curved space-times having the same or different signatures with respect to the Minkowski flat space-time. Therefore, based on the discussion displayed above, it is immediate to conclude that a NLPT which maps mutually the two space-times indicated above must necessarily exist in all cases considered here.

7 - CONCLUDING REMARKS

Physical insight on the class of transformations $\{P_s\}$ denoted here as special non-local point transformations (NLPT) emerges from the following two statements, represented respectively by: A) Proposition P₂ of THM.1 and B) the explicit realization obtained by the 4-velocity transformation laws (16) which follows in turn from Eqs.(24).

Let us briefly analyze the first one, i.e., in particular the fact that the set $\{P_S\}$ is endowed with the structure of a group. For this purpose, consider two arbitrary connected and time-oriented curved space-times $(\mathbf{Q}_{(i)}^4, g_{(i)})$ for $i = 1, 2$ and assume that the corresponding admissible subsets of (\mathbf{M}^4, η) , on which the same space-times are mapped by means of special NLPT, have a non-empty intersection with non-vanishing measure. The corresponding Jacobian are by assumption of the type (52) so that their product must necessarily belong to $\{P_S\}$ (Proposition P₂). The

conclusion is of outmost importance from the physical standpoint. Indeed, it implies that by means of two special NLPT it is possible to mutually map in each other *two*, in principle arbitrary, *curved space-times*. Therefore, the same theory can be applied in principle to the treatment of arbitrary curved space-times by means of the establishment of corresponding functional connections in terms of products of suitable special NLPT.

The validity of the second consideration is also of perspicuous evidence. In fact, the geometry of the transformed space-time (\mathbf{Q}^4, g) , which is represented by its metric tensor $g_{\mu\nu}(r)$ and the corresponding Riemann curvature tensor $R_{\sigma\mu\nu}^\rho(r)$, specifically arises because of suitable non-uniform 4-velocity transformation laws prescribed here. These also give rise to a related non-local point transformation (NLPT) occurring between the two space-times (\mathbf{Q}^4, g) and $(\mathbf{Q}'^4, g') \equiv (\mathbf{M}^4, \eta)$. In particular, in the case of the solution indicated above (see **Section 6**) for the transformation matrix $\mathcal{B}_\nu^\mu(r)$, it follows that the transformed 4-velocity has the following qualitative properties:

- Its time component, besides depending on the corresponding time-component of the Minkowski space-time, in general may carry also finite contributions which are linearly-dependent on all spatial components of the Minkowskian 4-velocity.
- The spatial components of the same 4-velocity depend linearly only on the corresponding spatial components the Minkowskian 4-velocity, and hence remain unaffected by its time component, i.e., its energy content in the Minkowski space-time.

In view of these considerations, we are now in position to draw the main conclusions.

This investigation carried out in this paper concerns basic theoretical issues and physical problems which have remained unsolved to date in the literature and whose solution obtained here is unprecedented in the literature and of critical importance in GR as well as relativistic theories such as Classical Electrodynamics, Kinetic Theory, Fluid and Magnetofluid Dynamics, Relativistic Quantum Mechanics. Indeed in this paper, a new approach to the standard formulation of GR (SF-GR) has been investigated based on the extension of the customary functional setting which lays at the basis of the same SF-GR. As an application, the Einstein's Teleparallel transformation problem (TT-problem) has been considered. This involves the construction of a suitable invertible coordinate transformation which maps in each other an arbitrary curved space-time (\mathbf{Q}^4, g) and the flat Minkowski space-time $(\mathbf{Q}'^4, g') \equiv (\mathbf{M}^4, \eta)$, both assumed to be parametrized in terms of Cartesian coordinates. As pointed out in THM.1, this requires necessarily the introduction of a new type of NLPT. Unlike local point transformations (LPT) traditionally adopted in SF-GR, these transformations have the distinctive property that the transformed space-time (\mathbf{Q}^4, g) may exhibit a non-vanishing Riemann curvature tensor. In particular, the new class of transformations is characterized by Jacobians with a characteristic non-gradient form.

The present outcome implies the extension of the traditional concept of point transformations and GR-reference frame adopted customarily in GR. Indeed, although the mathematical adequacy of SF-GR remains paramount in the framework of the LPT-functional setting, we have shown that there exist theoretical motivations based on actual physical problems, and in particular the TT-problem arising in the context of Einstein's theory of teleparallel gravity, which require a different functional setting. In fact, the solution of the same TT-problem requires changing both the functional setting as well as the notion of GR-reference frame usually adopted in GR. More precisely:

1. Based on the prescription of the coordinates and corresponding 4-velocity associated with each extended GR-frame, suitable phase-space transformations among them, denoted as NLPT-phase-space transformations, are introduced.
2. In particular, concerning the corresponding point transformations, these are identified with the special NLPT $\{P_S\}$ (see Eqs.(37)). These transformations reduce locally to LPT if the gradient conditions (29)-(30) apply.
3. The coordinate systems mapped in each other by means of a special NLPT belong, unlike in SF-GR, to two different space-times. In the case of the TT-problem these have been identified respectively with a generic curved space-time (\mathbf{Q}^4, g) and the flat Minkowski space-time $(\mathbf{Q}'^4, g') \equiv (\mathbf{M}^4, \eta)$, both represented in terms of Cartesian coordinates.
4. The class of special NLPT includes also coordinate transformations with map the Minkowski space-time onto a curved space-time characterized by a different signatures.

As shown here, the solution of the TT-problem rests purely on physical principles. In this regard in the present paper the following remarks have turned out to be crucial.

The first one is realized by Proposition #1, namely the fact that two different space-times, such as those occurring in the Einstein's TT-problem, namely (\mathbf{Q}^4, g) and $(\mathbf{Q}'^4, g') \equiv (\mathbf{M}^4, \eta)$, cannot be directly mapped in each other just

by means of a LPT. The second one, that general 4-velocity transformations of the form given by Eqs.(16) manifestly can always be introduced in which the Jacobian of the transformation is not of the gradient-form indicated by Eqs.(4) and (5). The third fundamental remark concerns the existence of NLTP. This is actually suggested by the Einstein equivalence principle itself, a principle which also lies at the heart of his approach to the TT-problem. Such a feature appears of critical importance. In fact, as shown here, it directly leads to the identification of the precise form of the NLPT which provides an explicit solution of the same TT-problem.

Finally, two characteristic aspects of the new (NLPT) transformations proposed here must be stressed. The first one is their non-locality, which appears both in their Lagrangian and Eulerian forms. This arises because of their non-local dependence with respect to 4-velocity. The second, and in turn related, one is due to the form of their Jacobians. In fact, in difference with the treatment of LPT, for NLPT the same ones are not identified with gradient operators. Nevertheless, since the Jacobians still are by assumption locally velocity-independent, tensor transformation laws can actually once again be recovered. These follow from the corresponding transformation equations which hold for the infinitesimal 4-position displacements and the corresponding 4-velocities.

ACKNOWLEDGMENTS

Work developed within the research projects of the Czech Science Foundation GAČR grant No. 14-07753P (C.C.) and Albert Einstein Center for Gravitation and Astrophysics, Czech Science Foundation No. 14-37086G (M.T.).

-
- [1] A. Einstein, *Die Feldgleichungen der Gravitation*, Sitzungsber. Preuss. Akad. Wiss. (Berlin), 844 (1915).
 - [2] J.L. Synge, A. Schild, *Tensor Calculus. first Dover Publications* 1978 edition. pp. 6–108 (1949).
 - [3] L.D. Landau and E.M. Lifschitz, *The classical theory of fields, Vol.2* (Addison-Wesley, N.Y., 1957).
 - [4] J.A. Wheeler, C. Misner, K.S. Thorne, *Gravitation*, W.H. Freeman & Co (1973).
 - [5] *ibid.*, pp. 85–86, §3.5 (1973).
 - [6] R.M. Wald, *General Relativity*, University of Chicago Press, 1st edition (1984).
 - [7] C. Cremaschini and M. Tassarotto, *Eur. Phys. J. Plus* **130**, 123 (2015).
 - [8] A. Einstein, *The Meaning of Relativity*, Princeton University Press (1945).
 - [9] A. Einstein, *Riemann-Geometrie mit Aufrechterhaltung des Begriffes des Fernparallelismus*, Preussische Akademie der Wissenschaften, Phys.-math. Klasse, Sitzungsberichte 217 (1928).
 - [10] A.O. Barvinsky, *Phys. Lett. B* **572**, 109 (2003).
 - [11] G. Calcagni and G. Nardelli, *Phys. Rev. D* **82**, 123518 (2010).
 - [12] A.O. Barvinsky, *Phys. Lett. B* **710**, 12 (2012).
 - [13] A.S. Koshelev, *Rom. J. Phys.* **57**, 894 (2012).
 - [14] G. Calcagni, L. Modesto and P. Nicolini, *Eur. Phys. J. C* **74**, 2999 (2014).
 - [15] B. Mashhoon, *Galaxies* **3**, 1 (2014).
 - [16] P. Wu and H. Yu, *Phys. Lett. B* **693**, 415 (2010).
 - [17] K. Bamba, C.Q. Geng, C.C. Lee and L.-W. Luo, *J. Cosmol. Astropart. Phys.* **01**, 021 (2011).
 - [18] T.P. Sotiriou, B. Li and J.D. Barrow, *Phys. Rev. D* **83**, 104030 (2011).
 - [19] E.T. Newman and A.I. Janis, *J. Math. Phys.* **6**, 915 (1965).
 - [20] E.T. Newman, E. Couch, K. Chinnapared, A. Exton, A. Prakash and R. Torrence, *J. Math. Phys.* **6**, 918 (1965).
 - [21] S.P. Drake and P. Szekeres, *Gen. Relativ. Gravit.* **32**, 445 (2000).
 - [22] C. Bambi and L. Modesto, *Phys. Lett. B* **721**, 329 (2013).
 - [23] B. Toshmatov, B. Ahmedov, A. Abdujabbarov and Z. Stuchlík, *Phys. Rev. D* **89**, 104017 (2014).
 - [24] L. Modesto and P. Nicolini, *Phys. Rev. D* **82**, 104035 (2010).
 - [25] U. Muench, F.W. Hehl, B.Mashhoon, *Phys. Lett. A* **271**, 8 (2000).
 - [26] B. Mashhoon, *Annalen der Physik* **523**, 226 (2011).
 - [27] C. Cremaschini and M. Tassarotto, *Eur. Phys. J. Plus* **126**, 42 (2011).
 - [28] C. Cremaschini and M. Tassarotto, *Eur. Phys. J. Plus* **126**, 63 (2011).
 - [29] C. Cremaschini and M. Tassarotto, *Eur. Phys. J. Plus* **127**, 4 (2012).
 - [30] C. Cremaschini and M. Tassarotto, *Phys. Rev. E* **87**, 032107 (2013).
 - [31] C. Cremaschini and M. Tassarotto, *Int. J. Mod. Phys. A* **28**, 1350086 (2013).
 - [32] C. Cremaschini and M. Tassarotto, *Eur. Phys. J. Plus* **129**, 247 (2014).
 - [33] C. Cremaschini and M. Tassarotto, *Eur. Phys. J. Plus* **130**, 166 (2015).
 - [34] A. Einstein, *Jahrbuch der Radioaktivität* **4**, 411 (1907).
 - [35] A. Einstein, *Annalen der Physik* **35**, 898 (1911).